

INDEPENDENCE OF HIGHER KUREPA HYPOTHESES

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ABSTRACT. We study the Generalized Kurepa Hypothesis introduced by Chang. We show that relative to the existence of an inaccessible cardinal the Gap- n -Kurepa hypothesis does not follow from the Gap- m -Kurepa hypothesis for m different from n . The use of an inaccessible is necessary for this result.

1. INTRODUCTION

In this paper we study the Generalized Kurepa Hypothesis introduced by Chang (see Chapter VII of [1]). We show that relative to the existence of an inaccessible cardinal the Gap- n -Kurepa hypothesis does not follow from the Gap- m -Kurepa hypothesis for m different from n . The use of an inaccessible is necessary for this result.

Definition 1.1. (a) For infinite cardinals $\lambda < \kappa$, a $\text{KH}(\kappa, \lambda)$ -family is a family \mathcal{F} of subsets of κ such that:

- (i) $\text{Card}(\mathcal{F}) \geq \kappa^+$,
- (ii) for all $x \in [\kappa]^\lambda$, $\text{Card}(\mathcal{F} \restriction x) \leq \lambda$, where $\mathcal{F} \restriction x = \{t \cap x : t \in \mathcal{F}\}$.

We say $\text{KH}(\kappa, \lambda)$ holds if such a family exists.

(b) For infinite cardinals $\lambda \leq \kappa$, a $\text{KH}(\kappa, < \lambda)$ -family is a family \mathcal{F} of subsets of κ such that:

- (i) $\text{Card}(\mathcal{F}) \geq \kappa^+$,
- (ii) for all $x \in [\kappa]^{<\lambda}$, $\text{Card}(\mathcal{F} \restriction x) \leq \text{Card}(x) + \aleph_0$.

We say $\text{KH}(\kappa, < \lambda)$ holds if such a family exists.

(c) Let $n \geq 1$, n finite. By the Gap- n -Kurepa hypothesis we mean the following statement: for all infinite cardinals λ , $\text{KH}(\lambda^{+n}, \lambda)$ holds.

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The following is well-known (see [1], Chapter VII, Theorems 3.2 and 3.3).

Theorem 1.2. (*Jensen*). *If $V = L$, then $\text{KH}(\kappa, < \lambda^+)$ (and hence $\text{KH}(\kappa, \lambda)$) holds for all infinite cardinals $\lambda < \kappa$, κ regular.*

In this paper we prove the following theorem.

Theorem 1.3. *Let $n \geq 1$. The following are equiconsistent:*

- (a) *There exists an inaccessible cardinal,*
- (b) *GCH + the Gap- m -Kurepa hypothesis holds for all $m \neq n$, but the Gap- n -Kurepa hypothesis fails.*

Remark 1.4. *Our proof shows that if $\lambda < \kappa$ are infinite cardinals, κ regular and $\text{KH}(\kappa, \lambda)$ fails, then κ^+ is inaccessible in L (see Lemma 3.1).*

Remark 1.5. *(b) of the above Theorem can be strengthened to the Gap- m -Kurepa hypothesis holds for all $m \neq n$, but $\text{KH}(\aleph_n, \aleph_0)$ fails (see Lemma 2.7).*

2. PROOF OF $\text{CON}(a)$ IMPLIES $\text{CON}(b)$

In this section we show that if there exists an inaccessible cardinal, then in a forcing extension of L , the Gap- m -Kurepa hypothesis holds for all $m \neq n$, but the Gap- n -Kurepa hypothesis fails, where $n \geq 1$ is a fixed natural number.

From now on assume that $V = L$, and let κ be an inaccessible cardinal. We consider two cases.

Case 1. $n = 1$.

Let $\mathbb{P} = \text{Col}(\omega_1, < \kappa)$ be the Levy collapse with countable conditions which converts κ into ω_2 , and let G be \mathbb{P} -generic over L .

Lemma 2.1. *The following hold in $L[G]$:*

- (a) *$\text{KH}(\aleph_1, \aleph_0)$ fails,*
- (b) *The Gap- m -Kurepa hypothesis holds for all $m \geq 2$.*

Proof. (a) is a well known result of Silver (see [7], or [2] Lemma 20.4).

(b) Let $m \geq 2$, and let λ be an infinite cardinal in $L[G]$. Let $\mu = (\lambda^{+m})^{L[G]}$. By Theorem 1.2, there is a $\text{KH}(\mu, \lambda)$ family \mathcal{F} in L . We show that it remains a $\text{KH}(\mu, \lambda)$ family in $L[G]$. Clearly $\text{Card}(\mathcal{F}) = \mu^{+L} = (\lambda^{+m+1})^{L[G]}$. Suppose $x \in ([\mu]^\lambda)^{L[G]}$.

Note that \mathbb{P} is κ -c.c. and ω_1 -closed, and in $L[G]$, κ becomes ω_2 . Thus it is easily seen that infinite sets in $L[G]$ are covered by sets of the same cardinality which belong to the ground model L , in particular there is a set $y \subseteq \mu$ in L such that $x \subseteq y$ and x and y have the same cardinality in $L[G]$. If $\lambda \neq \aleph_1$, then y has L -cardinality λ , hence in L , $\text{Card}(\mathcal{F} \restriction y) \leq \lambda$. It follows that in $L[G]$, $\text{Card}(\mathcal{F} \restriction x) \leq \text{Card}(\mathcal{F} \restriction y) \leq \lambda$. If $\lambda = \aleph_1$, then y has L -cardinality less than κ , hence in L , $\text{Card}(\mathcal{F} \restriction y) < \kappa$. It follows that in $L[G]$, $\text{Card}(\mathcal{F} \restriction y) \leq \aleph_1$, and hence in $L[G]$, $\text{Card}(\mathcal{F} \restriction x) \leq \text{Card}(\mathcal{F} \restriction y) \leq \aleph_1 = \lambda$. \square

Case 2. $n \geq 2$.

For each i , $0 < i < n$, fix an injection $J_i : [\omega_n]^{\leq \omega_i} \longrightarrow \omega_n$. Let $\mathbb{R} = \mathbb{P} \times \prod_{0 < i < n} \mathbb{Q}_i$, where the forcing notions \mathbb{P} and \mathbb{Q}_i , $0 < i < n$, are defined as follows.

$\mathbb{P} = \text{Col}(\omega_n, < \kappa)$ is the Levy collapse with conditions of size $< \omega_n$ which converts κ into ω_{n+1} .

\mathbb{Q}_i , $0 < i < n$, is the set of triples $p = (X_p, \mathcal{F}_p, g_p)$ such that:

(i-1) X_p is a subset of ω_n of size $\leq \omega_i$,

(i-2) \mathcal{F}_p is a subset of ${}^{X_p}2$ of size $\leq \omega_i$,

(i-3) g_p is a 1-1 function from a subset of κ into \mathcal{F}_p ,

(i-4) \mathcal{F}_p is ω_i -closed in the following sense: If $t \in {}^{X_p}2$ and $\langle X_\xi : \xi < \omega_{i-1} \rangle$ is a sequence of subsets of X_p such that for all $\xi < \omega_{i-1}$, $J_i(X_\xi) \in X_p$ and $t \restriction X_\xi \in \mathcal{F}_p \restriction X_\xi$, then there is $s \in \mathcal{F}_p$ such that $s \restriction X = t \restriction X$ and $s \restriction (X_p \setminus X) = 0 \restriction (X_p \setminus X)$ (=the zero function on $X_p \setminus X$), where $X = \bigcup_{\xi < \omega_{i-1}} X_\xi$.

For $p, q \in \mathbb{Q}_i$, let $p \leq q$ (p is an extension of q) iff:

(i-5) $X_p \supseteq X_q$,

(i-6) $\mathcal{F}_q = \mathcal{F}_p \restriction X_q$,

(i-7) $\text{dom}(g_p) \supseteq \text{dom}(g_q)$,

(i-8) for all $\alpha \in \text{dom}(g_q)$, $g_q(\alpha) = g_p(\alpha) \restriction X_q$.

We show that in the generic extension by \mathbb{R} , the Gap- m -Kurepa hypothesis holds for all $m \neq n$, but the Gap- n -Kurepa hypothesis fails.

Lemma 2.2. (a) \mathbb{P} is ω_n -closed,

(b) \mathbb{P} satisfies the κ -c.c.,

(c) Let $0 < i < n$. Then \mathbb{Q}_i is ω_{i+1} -closed modulo J_i in the following sense: If $\langle p_\xi : \xi < \lambda \rangle$, $\lambda \leq \omega_i$, is a descending sequence of conditions in \mathbb{Q}_i such that for all $\xi < \lambda$, $J_i(X_{p_\xi}) \in X_{p_{\xi+1}}$, then there is a condition $p \in \mathbb{Q}_i$ which extends all of the p_ξ 's, $\xi < \lambda$. Furthermore if $\lambda < \omega_i$, then p can be chosen to be the greatest lower bound of the p_ξ 's, $\xi < \lambda$.

(d) Let $0 < i < n$. Then \mathbb{Q}_i has the ω_{i+2} -c.c.

Proof. (a) and (b) are well known results of Levy (see [2], Lemma 20.4). We prove (c) and (d).

(c) Fix $0 < i < n$, and let $\langle p_\xi : \xi < \lambda \rangle$ be as above. To simplify the notation let $p_\xi = (X_\xi, \mathcal{F}_\xi, g_\xi)$, $\xi < \lambda$. We consider two cases.

Case 1. $\lambda < \omega_i$.

Let $p = (X, \mathcal{F}, g)$, where:

- $X = \bigcup_{\xi < \lambda} X_\xi$,
- \mathcal{F} is the least subset of ${}^X 2$ such that if $t \in {}^X 2$ and for all $\xi < \lambda$, $t \upharpoonright X_\xi \in \mathcal{F}_\xi$ then $t \in \mathcal{F}$, and \mathcal{F} is ω_i -closed in the sense of $(i-4)$,
- $\text{dom}(g) = \bigcup_{\xi < \lambda} \text{dom}(g_\xi)$,
- for all $\alpha \in \text{dom}(g)$, $g(\alpha) = \bigcup \{g_\xi(\alpha) : \xi < \lambda, \alpha \in \text{dom}(g_\xi)\}$.

It is easy to show that $p \in \mathbb{Q}_i$ and that p is the greatest lower bound for the sequence $\langle p_\xi : \xi < \lambda \rangle$.

Case 2. $\lambda = \omega_i$.

Let $p = (X, \mathcal{F}, g)$, where:

- $X = \bigcup_{\xi < \lambda} X_\xi$,
- $\text{dom}(g) = \bigcup_{\xi < \lambda} \text{dom}(g_\xi)$,
- for all $\alpha \in \text{dom}(g)$, $g(\alpha) = \bigcup \{g_\xi(\alpha) : \xi < \lambda, \alpha \in \text{dom}(g_\xi)\}$,

- \mathcal{F} is the least subset of ${}^X 2$ such that $\text{ran}(g) \cup \{t \restriction X_\xi \cup 0 \restriction (X \setminus X_\xi) : t \in X_\xi\} \subseteq \mathcal{F}$ and \mathcal{F} is ω_i -closed in the sense of $(i-4)$.

Then it is easy to show that $p \in \mathbb{Q}_i$ and that p is a lower bound for the sequence $\langle p_\xi : \xi < \lambda \rangle$.

(d) Fix $0 < i < n$. Suppose that \mathbb{Q}_i does not satisfy the ω_{i+2} -c.c. Let A be a maximal antichain in \mathbb{Q}_i of size $\geq \omega_{i+2}$. By a Δ -system argument we can assume that

- The sequence $\langle X_p : p \in A \rangle$ forms a Δ -system with root X .
- The sequence $\langle \text{dom}(g_p) : p \in A \rangle$ forms a Δ -system with root D .
- For all $p \neq q$ in A , $g_p \restriction D = g_q \restriction D$ and $\mathcal{F}_p \restriction X = \mathcal{F}_q \restriction X$.

Let θ be large regular, and let M be an elementary submodel of $H(\theta)$ of size ω_{i+1} which is closed under ω_i -sequences and such that $\mathbb{Q}_i, X, D, A \in M$. Pick $q \in A \setminus M$ and let $q \restriction M = (X_q \restriction M, \mathcal{F}_q \restriction M, g_q \restriction M)$, where:

- $X_q \restriction M = X_q \cap M$,
- $\mathcal{F}_q \restriction M = \{t \restriction (X_q \cap M) : t \in \mathcal{F}_q\}$,
- $\text{dom}(g_q \restriction M) = \text{dom}(g_q) \cap M$,
- for all $\alpha \in \text{dom}(g_q \restriction M)$, $(g_q \restriction M)(\alpha) = g_q(\alpha) \restriction (X_q \cap M)$.

Then $q \restriction M \in \mathbb{Q}_i \cap M$. Extend this condition to a condition $p \in \mathbb{Q}_i \cap M$ which extends an element $r \in A$. We show that p and q and hence r and q are compatible, which is impossible since $r, q \in A$.

Fix $s_0 \in \mathcal{F}_p, t_0 \in \mathcal{F}_q$. Define X, \mathcal{F} and g as follows:

- $X = X_p \cup X_q$,
- \mathcal{F} is the least subset of ${}^X 2$ such that $\{s \restriction X_p \cup t \restriction (X_q \setminus M) : s \in \mathcal{F}_p, t \in \mathcal{F}_q\} \subseteq \mathcal{F}$, and \mathcal{F} is ω_i -closed in the sense of $(i-4)$,
- $\text{dom}(g) = \text{dom}(g_p) \cup \text{dom}(g_q)$,
- $g(\alpha) = \begin{cases} g_p(\alpha) \restriction X_p \cup g_q(\alpha) \restriction (X_q \setminus M) & \text{if } \alpha \in \text{dom} g_q \cap M, \\ g_p(\alpha) \restriction X_p \cup t_0 \restriction (X_q \setminus M) & \text{if } \alpha \in \text{dom} g_p \setminus \text{dom} g_q, \\ g_q \restriction X_q \cup s_0 \restriction (X_p \setminus X_q) & \text{if } \alpha \in \text{dom} g_q \setminus M. \end{cases}$

Then $(X, \mathcal{F}, g) \in \mathbb{Q}_i$ and it extends both of p and q . \square

Let $K = G \times \prod_{0 < i < n} H_i$ be $\mathbb{R} = \mathbb{P} \times \prod_{0 < i < n} \mathbb{Q}_i$ generic over L . It follows from the above lemma that

- $\omega_i^{L[K]} = \omega_i^L$ for all $i \leq n$.
- $\omega_{n+1}^{L[K]} = \kappa^L$.

Lemma 2.3. *In $L[K]$, the Gap- m -Kurepa hypothesis holds for all $m \neq n$.*

Proof. First show that $\text{KH}(\aleph_n, \aleph_i)$ holds in $L[K]$, for all $0 < i < n$.

Claim 2.4. *Let $0 < i < n$. Forcing with \mathbb{Q}_i adds a family $\mathcal{F} \subseteq {}^{\omega_n}2$ such that*

- (a) $\text{Card}(\mathcal{F}) = \kappa$,
- (b) for all $X \in ([\omega_n]^{\omega_i})^L$, $\text{Card}(\mathcal{F} \restriction X) \leq \aleph_i$.

Proof. By Lemma 2.2, \mathbb{Q}_i is a cardinal preserving forcing notion. It is easy to prove the following (where H_i is assumed to be a \mathbb{Q}_i -generic filter over L):

- $\bigcup \{X_p : p \in H_i\} = \omega_n$,
- $\bigcup \{\text{dom}(g_p) : p \in H_i\} = \kappa$,
- for all $X \in ([\omega_n]^{\omega_i})^L$, there is some $p \in H_i$ with $X_p \supseteq X$,
- if $\alpha < \kappa$, then $g(\alpha) : \omega_n \rightarrow 2$, where

$$g(\alpha) = \bigcup \{g_p(\alpha) : p \in H_i, \alpha \in \text{dom}(g_p)\}$$

- if $\alpha < \beta < \kappa$, then $g(\alpha) \neq g(\beta)$.

Then $\mathcal{F} = \{g(\alpha) : \alpha < \kappa\}$ is as required. \square

Claim 2.5. *Infinite sets in $L[K]$ are covered by sets of the same cardinality which belong to the ground model L .*

Proof. It is easily seen that any infinite set of ordinals from $L[K]$ is covered by a set of ordinals of $L[G]$ of the same cardinality and that $L[K]$ and $L[G]$ have the same cardinals. On the other hand since \mathbb{P} is κ -c.c. and ω_n -closed and in $L[G]$, κ becomes ω_{n+1} , any infinite set of ordinals from $L[G]$ is covered by a set of ordinals of L of the same $L[G]$ -cardinality. The result follows immediately. \square

Now using the above Claim and the fact that $\omega_i^{L[K]} = \omega_i^L$, we can show that \mathcal{F} is in fact a $\text{KH}(\aleph_n, \aleph_i)$ -family in $L[K]$.

Next let λ be an infinite cardinal, $m \neq n$, and suppose $\mu = (\lambda^{+m})^{L[K]}$, $\mu \neq \aleph_n$. We show that $\text{KH}(\mu, \lambda)$ holds in $L[K]$.

Claim 2.6. $\text{KH}(\mu, \lambda)$ holds in $L[G]$.

Proof. If $\mu < \aleph_n$, the claim follows from the facts that $\text{KH}(\mu, \lambda)$ holds in L , $(\mu^+)^L = (\mu^+)^{L[G]}$ and L and $L[G]$ have the same μ -sequences. If $\mu > \aleph_n$, the claim follows exactly as in the proof of Lemma 2.1(b). \square

Using the facts that $L[G]$ and $L[K]$ have the same cardinals and any infinite set of ordinals from $L[K]$ is covered by a set of ordinals of $L[G]$ of the same cardinality, we can immediately conclude that $\text{KH}(\mu, \lambda)$ holds in $L[K]$. The Lemma follows. \square

Lemma 2.7. $\text{KH}(\aleph_n, \aleph_0)$ fails in $L[K]$.

Before going into the details of the proof of Lemma 2.7, we introduce some notions. Let λ be a regular cardinal, $\aleph_n < \lambda < \kappa$. Define the following forcing notions

$$\mathbb{P}_\lambda = \text{Col}(\omega_n, < \lambda),$$

$$\mathbb{Q}_{i,\lambda} = \text{the set of all } p \in \mathbb{Q}_i \text{ such that } \text{dom}(g_p) \subseteq \lambda,$$

$$\mathbb{R}_\lambda = \mathbb{P}_\lambda \times \prod_{0 < i < n} \mathbb{Q}_{i,\lambda}$$

Also let $K_\lambda = G_\lambda \times \prod_{0 < i < n} H_{i,\lambda}$ be \mathbb{R}_λ -generic over L . Define $\pi_\lambda : \mathbb{R} \longrightarrow \mathbb{R}_\lambda$ by

$$\pi_\lambda(\langle p, \langle (X_i, \mathcal{F}_i, g_i) : 0 < i < n \rangle \rangle) = \langle p \restriction \lambda, \langle (X_i, \mathcal{F}_i, g_i \restriction \lambda) : 0 < i < n \rangle \rangle$$

Claim 2.8. π_λ is a projection, i.e.

$$(a) \pi_\lambda(1_\mathbb{R}) = 1_{\mathbb{R}_\lambda},$$

$$(b) \pi_\lambda \text{ is order preserving,}$$

(c) if $r_0 \in \mathbb{R}_\lambda$, $r_1 \in \mathbb{R}$ and $r_0 \leq \pi_\lambda(r_1)$, then there is some $r \leq r_1$ in \mathbb{R} such that $\pi_\lambda(r) \leq r_0$.

Proof. (a) and (b) are trivial. We prove (c). Let $r_j = \langle p_j, \langle (X_{i,j}, \mathcal{F}_{i,j}, g_{i,j}) : 0 < i < n \rangle \rangle$, for $j = 0, 1$. Then $r = \langle p, \langle (X_i, \mathcal{F}_i, g_i) : 0 < i < n \rangle \rangle$ is as required, where:

- $p = p_0 \cup p_1 \restriction (\kappa \setminus \lambda)$,
- $X_i = X_{i,0}$,
- \mathcal{F}_i is the least subset of ${}^{X_i}2$ such that $\mathcal{F}_{i,0} \cup \{t \restriction X_{i,1} \cup 0 \restriction (X_{i,0} \setminus X_{i,1})\} \subseteq \mathcal{F}_i$, and \mathcal{F}_i is ω_i -closed in the sense of (i-4),
- $\text{dom} g_i = \text{dom} g_{i,0} \cup \text{dom} g_{i,1}$,

$$\bullet \ g_i(\alpha) = \begin{cases} g_{i,o}(\alpha) & \text{if } \alpha \in \text{dom} g_{i,0}, \\ g_{i,1}(\alpha) \upharpoonright X_{i,1} \cup 0 \upharpoonright (X_{i,0} \setminus X_{i,1}) & \text{if } \alpha \in \text{dom} g_{i,1} \setminus \lambda. \end{cases}$$

□

Let

$$(\mathbb{R} : \mathbb{R}_\lambda) = \{ \langle p, \langle (X_i, \mathcal{F}_i, g_i) : 0 < i < n \rangle \rangle \in \mathbb{R} : \pi_\lambda(\langle p, \langle (X_i, \mathcal{F}_i, g_i) : 0 < i < n \rangle \rangle) \in K_\lambda \}.$$

It follows from Lemma 2.2 (c) that

Claim 2.9. $(\mathbb{R} : \mathbb{R}_\lambda)$ is countably closed modulo the J_i 's, $0 < i < n$, in the following sense: if $\langle \langle p_m, \langle (X_{i,m}, \mathcal{F}_{i,m}, g_{i,m}) : 0 < i < n \rangle \rangle : m < \omega \rangle$ is a descending sequence of conditions in $(\mathbb{R} : \mathbb{R}_\lambda)$ such that for all $0 < i < n$ and $m < \omega$, $J_i(X_{i,m}) \in X_{i,m+1}$, then this sequence has a lower bound in $(\mathbb{R} : \mathbb{R}_\lambda)$.

Proof. For each $i, 0 < i < n$, the sequence $\langle (X_{i,m}, \mathcal{F}_{i,m}, g_{i,m}) : m < \omega \rangle$ is a descending sequence in \mathbb{Q}_i modulo J_i , thus by Lemma 2.2(c) it has a greatest lower bound $(X_i, \mathcal{F}_i, g_i)$. Let $r = \langle \bigcup_{m < \omega} p_m, \langle (X_i, \mathcal{F}_i, g_i) : 0 < i < n \rangle \rangle$. Then r is the greatest lower bound for the above sequence, and $\pi_\lambda(r)$ is a lower bound for the sequence $\langle \pi_\lambda(\langle p_m, \langle (X_{i,m}, \mathcal{F}_{i,m}, g_{i,m}) : 0 < i < n \rangle \rangle) : m < \omega \rangle$. Note that the projection π_λ just restricts the domain of functions involved in the condition to λ and thus we can easily show that:

- $\pi_\lambda(r)$ is in fact the greatest lower bound of the above sequence.
- If r' is compatible with all of $\langle p_m, \langle (X_{i,m}, \mathcal{F}_{i,m}, g_{i,m}) : 0 < i < n \rangle \rangle, m < \omega$, then r' is compatible with $\pi_\lambda(r)$.

It then follows from the maximality of K_λ that $\pi_\lambda(r) \in K_\lambda$, and hence $r \in (\mathbb{R} : \mathbb{R}_\lambda)$. Thus r is as required □

We are now ready to prove Lemma 2.7. Assume on the contrary that $\text{KH}(\aleph_n, \aleph_0)$ holds in $L[K]$. Suppose for simplicity that $1_{\mathbb{R}} \Vdash \dot{\mathcal{F}}$ is a $\text{KH}(\aleph_n, \aleph_0)$ -family $\dot{\mathcal{F}}$.

Let $\mathcal{F} = \dot{\mathcal{F}}[K]$, and let $A = \langle \mathcal{F} \upharpoonright X : X \in [\omega_n]^\omega \rangle$. Choose $\lambda < \kappa$ regular such that $A \in L[K_\lambda]$. Let $b \in \mathcal{F}$ be such that $b \notin L[K_\lambda]$.

From now on we work in $L[K_\lambda]$ and force with $(\mathbb{R} : \mathbb{R}_\lambda)$. Let \dot{b} be an $(\mathbb{R} : \mathbb{R}_\lambda)$ -name for b , and let $r_0 \in (\mathbb{R} : \mathbb{R}_\lambda)$, $r_0 = \langle p_0, \langle (X_{i,0}, F_{i,0}, g_{i,0}) : 0 < i < n \rangle \rangle$, be such that

$$r_0 \Vdash \ulcorner \dot{b} \in \dot{\mathcal{F}} \text{ and } \dot{b} \notin V \urcorner$$

It is easy to prove the following.

Claim 2.10. *For each $r \leq r_0$, $r = \langle p, \langle (X_i, F_i, g_i) : 0 < i < n \rangle \rangle$, there are two conditions $r_1 = \langle p_1, \langle (X_{i,1}, F_{i,1}, g_{i,1}) : 0 < i < n \rangle \rangle$, $r_2 = \langle p_2, \langle (X_{i,2}, F_{i,2}, g_{i,2}) : 0 < i < n \rangle \rangle$ and some $\xi < \omega_n$ such that:*

- (a) $r_1, r_2 \leq r$,
- (b) $J_i(X_i) \in X_{i,m}$ for all $0 < i < n$ and $m = 1, 2$,
- (c) $r_1 \Vdash \ulcorner \check{\xi} \in \dot{b} \urcorner$ iff $r_2 \Vdash \ulcorner \check{\xi} \notin \dot{b} \urcorner$. □

Using the above, we can construct a sequence $\langle r_s = \langle p_s, \langle (X_{i,s}, F_{i,s}, g_{i,s}) : 0 < i < n \rangle \rangle : s \in {}^{<\omega}2 \rangle$ of conditions in $(\mathbb{R} : \mathbb{R}_\lambda)$ and a sequence $\langle \xi_m : m < \omega \rangle$ of elements of ω_n such that the following hold:

- $r_{s*m} \leq r_s$, for each $s \in {}^{<\omega}2$ and $m < 2$,
- $J_i(X_{i,s}) \in X_{i,s*m}$ for each $s \in {}^{<\omega}2$, $m < 2$ and $0 < i < n$,
- $r_{s*0} \Vdash \ulcorner \check{\xi}_m \in \dot{b} \urcorner$ iff $r_{s*1} \Vdash \ulcorner \check{\xi}_m \notin \dot{b} \urcorner$, where m is the length of s .

Let $X = \{\xi_m : m < \omega\}$, and for each $f \in {}^\omega 2$, using Claim 2.9, let $r_f \in (\mathbb{R} : \mathbb{R}_\lambda)$ be an extension of all of the $r_{f \restriction m}$'s, $m < \omega$. For each f as above, we can find some $q_f \leq r_f$ and some $b_f \in L[K_\lambda]$ such that

$$q_f \Vdash \ulcorner \dot{b} \cap \check{X} = \check{b}_f \urcorner$$

Note that $\mathcal{F} \restriction X \supseteq \{b_f : f \in {}^\omega 2\}$ and for $f \neq g$ in ${}^\omega 2$, we have $b_f \neq b_g$, and hence $\mathcal{F} \restriction X$ must have size at least 2^{\aleph_0} which is in contradiction with our assumption.

It follows that $\text{KH}(\aleph_n, \aleph_0)$ fails in $L[K]$. This completes the proof of Lemma 2.7.

3. PROOF OF $\text{CON}(b)$ IMPLIES $\text{CON}(a)$

Now we show that if $n \geq 1$, and the Gap- n -Kurepa hypothesis fails, then there exists an inaccessible cardinal in L . In fact we will prove the following more general result.

Lemma 3.1. *Suppose that $\lambda < \kappa$ are infinite cardinals such that κ is regular, $\kappa^\lambda = \kappa$ and $\text{KH}(\kappa, \lambda)$ fails. Then κ^+ is an inaccessible cardinal in L .*

The rest of this section is devoted to the prove of the above lemma. Assume on the contrary that the lemma fails. Thus we can find $X \subseteq \kappa$ such that:

- V and $L[X]$ have the same cardinals up to κ^+ ,
- $([\kappa]^\lambda)^V = ([\kappa]^\lambda)^{L[X]}$.

It follows that a $\text{KH}(\kappa, \lambda)$ -family in $L[X]$ is a real $\text{KH}(\kappa, \lambda)$ -family, and hence $\text{KH}(\kappa, \lambda)$ fails in $L[X]$. The following lemma gives us the required contradiction.

Lemma 3.2. *Suppose that $V = L[X]$, where $X \subseteq \kappa$. Then $\text{KH}(\kappa, \lambda)$ holds.*

Proof. Our proof is very similar to the proof of Theorem 2 in [3]. We give it for completeness.

For each $x \in [\kappa]^\lambda$ let

$$M_x = \text{the smallest } M \prec L_\kappa[X] \text{ such that } x \cup \{x\} \cup (\lambda + 1) \subseteq M.$$

Let $\mathcal{F} = \{t \subseteq \kappa : \forall x \in [\kappa]^\lambda, t \cap x \in M_x\}$. We show that \mathcal{F} is a $\text{KH}(\kappa, \lambda)$ -family. It suffices to show that $\text{Card}(\mathcal{F}) \geq \kappa^+$. Suppose not. Let $C = \langle t_\nu : \nu < \kappa \rangle$ be an enumeration of \mathcal{F} definable in $L_{\kappa^+}[X]$. By recursion on $\nu < \kappa$, define a chain $\langle N_\nu : \nu < \kappa \rangle$ of elementary submodels of $L_{\kappa^+}[X]$ as follows:

$$N_0 = \text{the smallest } N \prec L_{\kappa^+}[X] \text{ such that } \lambda \in N \text{ and } N \cap \kappa \in \kappa,$$

$$N_{\nu+1} = \text{the smallest } N \prec L_{\kappa^+}[X] \text{ such that } N \cap \kappa \in \kappa \text{ and } N_\nu \cup \{N_\nu\} \subseteq N,$$

$$N_\delta = \bigcup_{\nu < \delta} N_\nu, \text{ if } \delta \text{ is a limit ordinal.}$$

For each $\nu < \kappa$ set $\alpha_\nu = N_\nu \cap \kappa$. Using the condensation lemma for $L[X]$, we obtain an ordinal β_ν and an isomorphism σ_ν such that

$$\sigma_\nu : \langle N_\nu, \in, N_\nu \cap X \rangle \simeq \langle L_{\beta_\nu}[X \cap \alpha_\nu], \in, X \cap \alpha_\nu \rangle.$$

Then:

- $\alpha_\nu < \beta_\nu < \alpha_{\nu+1}$,
- $\sigma_\nu(\kappa) = \alpha_\nu$,
- $\sigma_\nu(X) = X \cap \alpha_\nu$,
- $\sigma_\nu \upharpoonright \alpha_\nu = id \upharpoonright \alpha_\nu$,

- $L_{\beta_\nu}[X \cap \alpha_\nu] \models \ulcorner \alpha_\nu \text{ is a regular cardinal, and } \alpha_\nu \text{ is the largest cardinal } \urcorner$.

Let $t = \{\beta_\nu : \beta_\nu \notin t_\nu\}$. Clearly $t \neq t_\nu$ for all $\nu < \kappa$, and hence $t \notin \mathcal{F}$. Let $x \in [\kappa]^\lambda$ be such that:

- $t \cap x \notin M_x$,
- $\alpha = \sup(x)$ is minimal.

It follows that $t \cap x$ is cofinal in α , and hence $\alpha = \alpha_\eta$ for some $\eta < \kappa$. We have

$$t \cap x = \{\beta_\nu \in x : \beta_\nu < \alpha_\eta \text{ and } \beta_\nu \notin t_\nu \cap \alpha_\eta\}$$

and thus $t \cap x$ is definable from x , $\langle \beta_\nu : \nu < \eta \rangle$ and $\langle t_\nu \cap \alpha_\eta : \nu < \eta \rangle$. It is clear that:

- $x \in M_x$,
- $\langle \beta_\nu : \nu < \eta \rangle$ is definable in $L_{\beta_\eta}[X \cap \alpha_\eta]$.
- $\sigma_\eta(C) = \langle t_\nu \cap \alpha_\eta : \nu < \eta \rangle$, and hence $\langle t_\nu \cap \alpha_\eta : \nu < \eta \rangle$ is definable in $L_{\beta_\eta}[X \cap \alpha_\eta]$.

Clearly $X \cap \alpha_\eta \in M_x$. We show that $\beta_\eta \in M_x$. It will follow that $t \cap x \in M_x$ which is a contradiction. The proof is in a sequence of claims. Let $M = M_x$.

Claim 3.3. $\mathcal{P}(\alpha_\eta) \cap M \not\subseteq L_{\beta_\eta}[X \cap \alpha_\eta]$.

Proof. Suppose not. Since $cf(\alpha_\eta) = cf(x) \leq \lambda < \alpha_\eta$, there is $a \in M$ such that $a \subseteq \alpha_\eta$ is cofinal in α_η and has order type less than α_η . Then $a \in L_{\beta_\eta}[X \cap \alpha_\eta]$, and hence α_η is not a regular cardinal in $L_{\beta_\eta}[X \cap \alpha_\eta]$. A contradiction. \square

For $l < \nu < \kappa$ set:

- $\alpha^{(\nu)} = \langle \alpha_\iota : \iota \leq \nu \rangle$,
- $\beta^{(\nu)} = \langle \beta_\iota : \iota \leq \nu \rangle$,
- $\sigma_{\iota\nu} = \sigma_\nu \sigma_\iota^{-1} : \langle L_{\beta_\iota}[X \cap \alpha_\iota], \in, X \cap \alpha_\iota \rangle \longrightarrow \langle L_{\beta_\nu}[X \cap \alpha_\nu], \in, X \cap \alpha_\nu \rangle$,
- $\sigma^{(\nu)} = \langle \sigma_{\iota\nu} : \iota < \tau \leq \nu \rangle$.

Claim 3.4. $\nu \in M \cap \eta$ implies $\alpha^{(\nu)}, \beta^{(\nu)}, \sigma^{(\nu)} \in M$.

Proof. First note that $\alpha_\nu \in M$ implies $\alpha^{(\nu)} \in M$, since $\langle \alpha_\iota : \iota < \nu \rangle$ is definable from $L_{\beta_\nu}[X \cap \alpha_\nu]$ the way $\langle \alpha_\iota : \iota < \kappa \rangle$ was defined from $L_{\kappa^+}[X]$. It follows that $\nu \in M \cap \eta$ implies $\alpha^{(\nu)} \in M$, since there is $\tau, \nu \leq \tau < \eta$ such that $\alpha_\tau \in M$ and $\alpha_\nu = \alpha^\tau(\nu) \in M$. By similar arguments $\nu \in M \cap \eta$ implies $\beta^{(\nu)}, \sigma^{(\nu)} \in M$. \square

We note that

$$\langle \langle L_{\beta_\iota}[X \cap \alpha_\iota], \in, X \cap \alpha_\iota \rangle_{\iota < \eta}, \langle \sigma_{\iota\nu} \rangle_{\iota < \nu < \eta} \rangle$$

is a directed system of elementary embeddings, and if

$$\langle \langle U, E, Y \rangle, \langle g_\iota \rangle_{\iota < \eta} \rangle$$

is its direct limit, then:

- $\langle U, E, Y \rangle \simeq \langle L_{\beta_\eta}[X \cap \alpha_\eta], \in, X \cap \alpha_\eta \rangle$,
- $g_\iota : \langle L_{\beta_\iota}[X \cap \alpha_\iota], \in, X \cap \alpha_\iota \rangle \longrightarrow \langle U, E, Y \rangle$,
- If $f : \langle U, E, Y \rangle \simeq \langle L_{\beta_\eta}[X \cap \alpha_\eta], \in, X \cap \alpha_\eta \rangle$, then $\sigma_{\iota\eta} = fg_\iota$.

Now let $\pi : \langle M, \in, M \cap X \rangle \simeq \langle L_\delta[\tilde{X}], \in, \tilde{X} \rangle$, where $\tilde{X} = \pi[M \cap X]$. Let

- $\tilde{\alpha}^{(\nu)} = \pi(\alpha^{(\nu)})$,
- $\tilde{\beta}^{(\nu)} = \pi(\beta^{(\nu)})$,
- $\tilde{\sigma}^{(\nu)} = \pi(\sigma^{(\nu)})$,
- $\tilde{\alpha} = \bigcup_{\nu \in M \cap \eta} \tilde{\alpha}^{(\nu)}$,
- $\tilde{\beta} = \bigcup_{\nu \in M \cap \eta} \tilde{\beta}^{(\nu)}$,
- $\tilde{\sigma} = \bigcup_{\nu \in M \cap \eta} \tilde{\sigma}^{(\nu)}$,

and

- $\tilde{\alpha}_\iota = \pi(\alpha_{\pi^{-1}(\iota)})$,
- $\tilde{\beta}_\iota = \pi(\beta_{\pi^{-1}(\iota)})$,
- $\tilde{\sigma}_{\iota\nu} = \pi(\sigma_{\pi^{-1}(\iota), \pi^{-1}(\nu)})$.

Now

$$\langle \langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \in, \tilde{X} \cap \tilde{\alpha}_\iota \rangle_{\iota < \pi(\eta)}, \langle \tilde{\sigma}_{\iota\nu} \rangle_{\iota < \nu < \pi(\eta)} \rangle$$

is a directed system of elementary embeddings. Let

$$\langle \langle \tilde{U}, \tilde{E}, \tilde{Y} \rangle, \langle \tilde{g}_\iota \rangle_{\iota < \pi(\eta)} \rangle$$

be its direct limit. Then

- $\tilde{g}_\iota : \langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \in, \tilde{X} \cap \tilde{\alpha}_\iota \rangle \longrightarrow \langle \tilde{U}, \tilde{E}, \tilde{Y} \rangle$,

- There is an elementary embedding h such that the following diagram is commutative

$$\begin{array}{ccc}
\langle L_{\beta_{\pi^{-1}(\iota)}}[X \cap \alpha_{\pi^{-1}(\iota)}], \in, X \cap \alpha_{\pi^{-1}(\iota)} \rangle & \xrightarrow{g_{\pi^{-1}(\iota)}} & \langle U, E, Y \rangle \\
\pi^{-1} \uparrow & & \uparrow h \\
\langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \in, \tilde{X} \cap \tilde{\alpha}_\iota \rangle & \xrightarrow{\tilde{g}_\iota} & \langle \tilde{U}, \tilde{E}, \tilde{Y} \rangle
\end{array}$$

It follows that $\langle \tilde{U}, \tilde{E} \rangle$ is well founded. Let

$$\tilde{f} : \langle \tilde{U}, \tilde{E}, \tilde{Y} \rangle \simeq \langle L_{\tilde{\beta}}[\tilde{X}], \in, \tilde{X} \rangle.$$

Also let

- $\bar{\sigma}_\iota = \tilde{f}\tilde{g}_\iota : \langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \in, \tilde{X} \cap \tilde{\alpha}_\iota \rangle \longrightarrow \langle L_{\tilde{\beta}}[\tilde{X}], \in, \tilde{X} \rangle,$
- $\pi^* = f h \tilde{f}^{-1} : \langle L_{\tilde{\beta}}[\tilde{X}], \in, \tilde{X} \rangle \longrightarrow \langle L_{\beta_\eta}[X \cap \alpha_\eta], \in, X \cap \alpha_\eta \rangle.$

Then $\tilde{\sigma}_{\iota\tau} = \bar{\sigma}_\tau^{-1} \bar{\sigma}_\iota$ for $\iota < \tau < \pi(\eta)$, and the following diagram is commutative

$$\begin{array}{ccc}
\langle L_{\beta_{\pi^{-1}(\iota)}}[X \cap \alpha_{\pi^{-1}(\iota)}], \in, X \cap \alpha_{\pi^{-1}(\iota)} \rangle & \xrightarrow{\sigma_{\pi^{-1}(\iota), \eta}} & \langle L_{\beta_\eta}[X \cap \alpha_\eta], \in, X \cap \alpha_\eta \rangle \\
\pi^{-1} \uparrow & & \uparrow \pi^* \\
\langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \in, \tilde{X} \cap \tilde{\alpha}_\iota \rangle & \xrightarrow{\bar{\sigma}_\iota} & \langle L_{\tilde{\beta}}[\tilde{X}], \in, \tilde{X} \rangle
\end{array}$$

Let $\bar{\alpha}$ be such that $L_{\tilde{\beta}}[\tilde{X}] \models \ulcorner \bar{\alpha} \text{ is the largest cardinal } \urcorner$.

Claim 3.5. (a) $\pi(\alpha_\eta) = \bar{\alpha}$,

(b) $\pi^*(\bar{\alpha}) = \alpha_\eta$,

(c) $\pi^* \upharpoonright \bar{\alpha} = id \upharpoonright \bar{\alpha}$.

Proof. (a) follows easily from the facts that $\bar{\alpha} = \sup_{\iota < \eta} \tilde{\alpha}_\iota$, $\alpha_\eta = \sup_{\iota \in M \cap \eta} \alpha_\iota$ and $\pi^{-1}(\tilde{\alpha}_\iota) = \alpha_{\pi^{-1}(\iota)}$. (b) follows from the choice of $\bar{\alpha}$ and the elementarity of π^* . (c) is trivial, as $\bar{\alpha} \subseteq L_{\tilde{\beta}}[\tilde{X}]$. \square

Next we have

Claim 3.6. If $a \subseteq \bar{\alpha}$ and $a \in L_{\tilde{\beta}}[\tilde{X}] \cap L_\delta[\tilde{X}]$, then $\pi^*(a) = \pi^{-1}(a)$.

Proof. Since $a \subseteq \bar{\alpha}$, $\pi^*(a)$, $\pi^{-1}(a) \subseteq \alpha_\eta$, and hence $\pi^*(a) = \bigcup_{\nu \in M \cap \eta} \pi^*(a) \cap \nu = \bigcup_{\nu < \pi(\eta)} \pi^*(a \cap \nu)$. \square

$\nu) \stackrel{\text{claim 3.5}}{=} \bigcup_{\nu < \pi(\eta)} \pi^{-1}(a \cap \nu) = \pi^{-1}(a).$

Claim 3.7. $\delta > \bar{\beta}$.

Proof. Suppose not. Then $\delta \leq \bar{\beta}$ and $\pi^*\pi$ maps M into $L_{\beta_\eta}[X \cap \alpha_\eta]$, and by claim 3.6, $\pi^*\pi(a) = a$ for $a \subseteq \alpha_\eta$, $a \in M$. It follows that $\mathcal{P}(\alpha_\eta) \cap M \subseteq L_{\beta_\eta}[X \cap \alpha_\eta]$, which is in contradiction with claim 3.3. \square

It follows that $\bar{\beta} \in L_\delta[\tilde{X}]$ and hence $\tilde{\beta} = \langle \tilde{\beta}_\iota : \iota < \pi(\eta) \rangle \in L_\delta[\tilde{X}]$, since $\tilde{\beta}$ is definable from $L_{\bar{\beta}}[\tilde{X}]$ as $\langle \tilde{\beta}_\iota : \iota < \kappa \rangle$ was defined from $L_{\kappa^+}[X]$. Similarly $\tilde{\sigma} = \langle \tilde{\sigma}_{\iota, \nu} : \iota < \nu < \pi(\eta) \rangle \in L_\delta[\tilde{X}]$. It is easily seen that

Claim 3.8. (a) $\pi^{-1}(\tilde{\alpha}) = \langle \alpha_\iota : \iota < \eta \rangle$,

(b) $\pi^{-1}(\tilde{\beta}) = \langle \beta_\iota : \iota < \eta \rangle$,

(c) $\pi^{-1}(\tilde{\sigma}) = \langle \sigma_{\iota, \nu} : \iota < \nu < \eta \rangle$. \square

Now note that:

- $L_{\bar{\beta}}[\tilde{X}]$ is the direct limit of $L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota]$, $\tilde{\sigma}_{\iota, \nu}, \iota < \nu < \pi(\eta)$,
- $\pi^{-1}[\tilde{X}] = X \cap \alpha_\eta$,
- $\pi^{-1}[\tilde{X} \cap \tilde{\alpha}_\iota] = X \cap \alpha_\iota$,

and hence by elementarity of π^{-1} , $L_{\pi^{-1}(\bar{\beta})}[X \cap \alpha_\eta]$ is the direct limit of $L_{\beta_\iota}[X \cap \alpha_\iota]$, $\sigma_{\iota, \nu}, \iota < \nu < \eta$.

It follows that $\pi^{-1}(\bar{\beta}) = \beta_\eta \in M$. We are done. \square

4. OPEN PROBLEMS

We close the paper with some remarks and open problems.

By the results of Vaught, Chang, Jensen (see [1], Chapter VIII) and Silver (see [7]), it is consistent, relative to the existence of an inaccessible cardinal, to have the Gap- n -transfer principle with the failure of the gap- $(n+1)$ -transfer principle for $n = 1$. The answer is unknown for $n > 1$.

Question 4.1. *Let $n > 1$. Is it consistent to have the Gap- n -transfer principle with the failure of the Gap- $(n+1)$ -transfer principle?*

Another related question is

Question 4.2. *Let $n > 1$. Is it consistent to have (κ, n) –morasses for each uncountable regular κ , but no $(\omega_1, n + 1)$ –morasses?*

Remark 4.3. *Assuming the existence of large cardinals, it is possible to build a model of set theory in which there exists a $(\kappa, 1)$ –morass for each uncountable regular κ , but there are no $(\omega_1, 2)$ –morasses.*

In the literature the canonical counter-example to the Gap–1–transfer principle is the non-existence of Special Aronszajn trees (see [5]). T. Raesch, in his dissertation (see [6]), showed that this principle can fail in the presence of such trees. On the other hand the canonical counter-example to the Gap–2–transfer principle is the non-existence of Kurepa trees (see [7]). Inspired by the work of Raesch, Jensen produced, relative to the existence of a Mahlo cardinal, a model in which the Gap–2–transfer principle fails, while the Gap–1–Kurepa hypothesis holds (see [4]). However the following is open.

Question 4.4. *Is it consistent relative to an inaccessible cardinal to have the Gap–1–Kurepa Hypothesis but a failure of the Gap –2–transfer principle?*

Remark 4.5. *It is possible to show that the existence of an $(\omega_2, 1)$ –morasses implies $\text{KH}(\aleph_2, < \aleph_2)$. Thus in our model, for $n = 2$, the Gap–1–Kurepa hypothesis holds, while in it there are no $(\omega_2, 1)$ –morasses.*

Question 4.6. *Let $n > 1$. Is it consistent with GCH to have $\text{KH}(\aleph_n, \aleph_0)$ but not $\text{KH}(\aleph_n, \aleph_1)$?*

Question 4.7. *Let $n > 1$. Is it consistent with GCH to have $\text{KH}(\aleph_n, \aleph_i)$ for all $i < n$, but not $\text{KH}(\aleph_n, < \aleph_n)$?*

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